Math 5C Discussion Problems 1: Selected Solutions

Line Integrals

1. For each of the following, compute $\int_C f\,ds$.

   (a) $f(x,y) = 2x - y$, C parametrized by $\mathbf{r}(t) = (e^t + 1, e^t - 2)$, $0 \le t \le \ln 2$

   Solution. Since $ds = \|\mathbf{r}'(t)\|\,dt = \|(e^t, e^t)\|\,dt = e^t\sqrt{2}\,dt$,

   $$\int_C f\,ds = \int_0^{\ln 2} (e^t + 4)e^t\sqrt{2}\,dt = \sqrt{2}\left(\frac{e^{2t}}{2} + 4e^t\right)\Bigg|_0^{\ln 2} = \frac{11\sqrt{2}}{2} \qquad \square$$

   (f) $f(x,y,z) = (x+y+z)/(x^2+y^2+z^2)$, C is the straight-line path from $(1,1,1)$ to $(2,2,2)$

   Solution. Parametrize C as $\mathbf{r}(t) = (t,t,t)$, $1 \le t \le 2$. Then $ds = \|\mathbf{r}'\|\,dt = \sqrt{3}\,dt$.

   $$\int_C f\,ds = \int_1^2 \frac{\sqrt{3}}{t}\,dt = \sqrt{3}\ln 2 \qquad \square$$

2. Let C be the circle of radius 4 centered at the origin in $\mathbb{R}^2$. Without integrating, evaluate

   $$\int_C \exp(x^2 + y^2)\,ds.$$

   Solution. On C, $x^2 + y^2 = 16$. So $\int_C \exp(x^2+y^2)\,ds = e^{16}\int_C ds = 8\pi e^{16}$. $\square$

3. For each of the following, compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

   (a) $\mathbf{F} = (y^2, -x^2)$, C is the part of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$

   Solution. Parametrize C as $\mathbf{r}(t) = (t, t^2)$, $-1 \le t \le 1$. Then on C, $\mathbf{F} = (t^4, -t^2)$ so

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 (t^4, -t^2)\cdot(1, 2t)\,dt = \int_{-1}^1 (t^4 - 2t^3)\,dt = \frac{2}{5} \qquad \square$$

5. Evaluate $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$, where C is the circle $x^2+y^2 = 1$, oreinted counterclockwise.

   Solution. Parametrize $\mathbf{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Then

   $$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} dt = 2\pi \qquad \square$$

Double Integrals

1. Evaluate the following integrals.

   (a) $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx$

   Solution. Compute: $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 y\cos(x^2)\Big|_0^x\,dx = \int_0^1 x\cos(x^2)\,dx = \frac{\sin 1}{2}$ □

3. Evaluate the following integrals. Consider interchanging the order of integration.

   (b) $\int_0^1 \int_{y^{1/3}}^1 e^{x^4}\,dx\,dy$

   Solution. The other order of integration is $0 \le x \le 1$, $0 \le y \le x^3$ (sketch a picture to see this).
   $\int_0^1 \int_{y^{1/3}}^1 e^{x^4}\,dx\,dy = \int_0^1 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^1 x^3 e^{x^4}\,dx = \frac{e-1}{4}$ □

4. Suppose that $\int_0^1 f(x)\,dx = A$ and $\int_0^1 g(y)\,dy = B$. What is $\iint_{[0,1]^2} f(x)g(y)\,dA$?

   Solution.
   $$\iint_{[0,1]^2} f(x)g(y)\,dA = \int_0^1 \int_0^1 f(x)\,g(y)\,dx\,dy = \int_0^1 g(y)\left(\int_0^1 f(x)\,dx\right)\,dy$$
   The quantity in parentheses is a *number*, so it can pulled out of the y integral.
   $$\iint_{[0,1]^2} f(x)g(y)\,dA = \int_0^1 g(y)\left(\int_0^1 f(x)\,dx\right)\,dy = \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(y)\,dy\right)$$ □

7. For each of the following, evaluate the integral. Consider polar coordinates.

   (b) $\iint_D (x^2+y^2)^{3/4}\,dA$, where D is the disk centered at the origin with radius 4

   Solution. In polar, D is given by $0 \le \theta \le 2\pi$ and $0 \le r \le 4$.
   $$\iint_D (x^2+y^2)^{3/4}\,dA = \int_0^{2\pi}\int_0^4 r^{5/2}\,dr\,d\theta = \frac{512\pi}{7}$$ □

   (f) $\iint_D (x^2+y^2)^{-1/2}\,dA$, where D is the unit disk centered at $(1,0)$

   Solution. Describing the circle: from $(x-1)^2+y^2=1$ we get $x^2+y^2 \le 2x$, or $r = 2\cos\theta$ in polar coordinates. Drawing a sketch helps to see how to use this: The disk is described by $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$.
   $$\iint_D (x^2+y^2)^{-1/2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} dr\,d\theta = 4$$ □

Surfaces and Their Integrals

1. For the surface parametrized by

$$x = \cos v \sin u, \qquad y = \sin v \sin u, \qquad z = \cos u$$

with $0 \le u \le \pi$ and $0 \le v \le 2\pi$, compute an expression for the unit normal vector in terms of u and v as well as the surface area. Identify the surface.

Solution. Writing $\mathbf{r}(u,v) = (\cos v \sin u, \sin v \sin u, \cos u)$ gives $\mathbf{r}_u = (\cos v \cos u, \sin v \cos u, -\sin u)$ and $\mathbf{r}_v = (-\sin v \sin u, \cos v \sin u, 0)$, so the unit normal is

$$\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = (\cos v \sin u, \sin v \sin u, \cos u).$$

The surface is a sphere. □

5. Let S be the surface with $x^2 + y^2 \le 1$ and $z = x^2 + y^2$.

(a) Find the area of S.

(b) Evaluate $\iint_S z \, d\sigma$.

(c) Evaluate $\iint_S (x, y, z) \cdot d\mathbf{A}$.

Solution. Parametrize S as $\mathbf{r}(u,v) = (u \cos v, u \sin v, u^2)$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$. Then

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= u(-2u \cos v, -2u \sin v, 1) \\ \|\mathbf{r}_u \times \mathbf{r}_v\| &= u\sqrt{1 + 4u^2} \end{aligned}$$

So $d\sigma = u\sqrt{1+4u^2}\, du\, dv$ and $d\mathbf{A} = u(-2u \cos v, -2u \sin v, 1)\, du\, dv$ (this makes the surface oriented upward).

$$\begin{aligned} \text{area}(S) = \iint_S d\sigma &= \int_0^{2\pi} \int_0^1 u\sqrt{1+4u^2}\, du\, dv = \frac{\pi}{6}\left(5\sqrt{5} - 1\right) \\ \iint_S z \, d\sigma &= \int_0^{2\pi} \int_0^1 u^3\sqrt{1+4u^2}\, du\, dv = \frac{\pi}{60}\left(25\sqrt{5} + 1\right) \\ \iint_S (x, y, z) \cdot d\mathbf{A} &= \int_0^{2\pi} \int_0^1 -u^3 \, du\, dv = -\frac{\pi}{2} \end{aligned}$$

□

9. Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ with $1 \le z \le 2$ and downward-pointing normal vector. Compute

$$\iint_S (x^2, y^2, z^2) \cdot d\mathbf{A}$$

Solution. Parametrize S as $r(u,v) = (u \cos v, u \sin v, u)$, where $1 \le u \le 2$ and $0 \le v \le 2\pi$. Then $r_u \times r_v = (-u \cos v, -u \sin v, u)$. However, this has the wrong orientation (we want a negative z component). To obtain the correct answer, we multiply by -1:

$$\begin{aligned} \iint_S (x^2, y^2, z^2) \cdot d\mathbf{A} &= -\int_0^{2\pi} \int_1^2 (u^2 \cos^2 v, u^2 \sin^2 v, u^2) \cdot (-u \cos v, -u \sin v, u)\, du\, dv \\ &= \int_0^{2\pi} \int_1^2 u^3(\cos^3 v + \sin^3 v - 1)\, du\, dv = -\frac{15\pi}{2} \end{aligned}$$

□

Triple Integrals

1. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.

   (c) Evaluate $\iiint_R z\,dV$.

   Solution. $\iiint_R z\,dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} z\,dz\,dy\,dx = \frac{1}{8}$ □

4. Integrate $\sqrt{x^2 + y^2 + z^2}\,e^{-(x^2+y^2+z^2)}$ over the unit ball centered at the origin.

   Solution. In spherical,

$$\iiint_B \sqrt{x^2 + y^2 + z^2}\,e^{-(x^2+y^2+z^2)}\,dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 e^{-\rho^2} \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\left(1 - 2e^{-4}\right)$$

□

7. Evaluate $\iiint_{\mathbb{R}^3} e^{-z^2} \frac{\sqrt{x^2 + y^2}}{1 + (x^2 + y^2)^{3/2}}\,dV$.

   Solution. In cylindrical,

$$\begin{aligned}
\iiint_{\mathbb{R}^3} e^{-z^2} \frac{\sqrt{x^2 + y^2}}{1 + (x^2 + y^2)^{3/2}}\,dV &= \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{r^2 e^{-z^2}}{1 + r^3}\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{r^2\sqrt{\pi}}{1 + r^3}\,dr\,d\theta \\
&= \frac{2\pi\sqrt{\pi}}{3} \ln(1 + r^3)\Big|_0^{\infty} \\
&= \infty
\end{aligned}$$

□